The Rational Recursive Sequence

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***ABSTRACT*:**

In this paper we investigate the global attractor of all positive solution of the rational recursive sequence

$y_{n+1} = \frac{by_n}{1-ay_n}$ $, n = 0,1,2, \ldots..$ *where* $a, b \in (0, \infty)$, $b > 1$

.

Keywords: Rational recursive sequence, locally stable, locally asymptotically stable, Global attractor, globally asymptotically stable.

المتتالية التكرارية الكسرية

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المستخلص:

في هذه البحث سوف نقوم بالتحقيق في عامل الجذب العالمي لجميع الحلول الموجبة للمتتالية التكرارية الكسرية

$$y_{n+1} = \frac{by_n}{1-ay_n} \quad , n = 0,1,2, \dots.. \quad \text{where } a,b \in (0,\infty) \ , \ b > 1$$

الكلمات المفتاحية: المتتالية التكرارية الكسرية، الاستقرار المحلي، الاستقرار غير المتناهي المحلي، الجاذب العالمي، الاستقرار غير المتناهي العالمي.

1. INTRODUCTION

Recently, many researchers have studied the *rational recursive sequence for both linear and nonlinear difference equation.* (Wantong, Yanhong, & Youhui, 2005) *examined the global attractivity of the nonlinear difference equation*

$x_{n+1} = \frac{a+bx_n}{A+x_{n-k}}, \quad n = 0,1,2 \dots..$ *where* $a, b, A \in (0,\infty), k$ *is a positive integer and the initial conditions* $x, \dots.., x_{-1}$ *and* x_0 *are arbitrary positive numbers,* (Koci'C, V.L.,, & Ladas, G, 1992) *established the result for global attractivity in the nonlinear delay difference equation*

$x_{n+1} = x_n f(x_{n-k}), \quad n = 0,1,2, \dots.$ *where* k *is a nonnegative integer, In* (Zhang, SHI, & GAI, 2001) *the global attractivity of all positive solutions of a rational recursive sequence was obtained* $x_{n+1} = \frac{a+bx_n^2}{1+x_{n-1}^2}$ *for* $n = 0,1,2, \dots \dots$, (Camouzis, Ladas, Rodrigues, & North shied, 1994) *studied the behavior of solution of the* $x_{n+1} = {(\beta x_n^2)}\Big/_{(1+x_{n-1}^2)}$ *where* β *is a positive constant and the initial condition* x_{-1} *and* x_0 *are arbitrary positive numbers, in* (Aboutaleb, EL-Sayed, & Hamza, 2001) *studied the global behavior of the recursive sequence* $x_{n+1} = \frac{a-bx_n}{A+x_{n-1}}$ *for* $n = 0,1,\ldots$ *where* a, b, A *are nonnegative real numbers and obtained sufficient conditions for the global attractive of the positive equilibria, and* (Clemente, Donten-Bury, Mazowiecki, & Pilipczuk, 2023)*studied the class of rational recursive sequences over the rational numbers.*

In this paper Consider the rational recursive sequence

$$y_{n+1} = \frac{by_n}{1-ay_n} \qquad ,n = 0,1,2,\ldots.. \tag{1}$$

Where

$$a, b \in (0,\infty)\ ,\ b > 1 \tag{2}$$

If a_0 *and* a_{-1} *are has given nonnegative numbers, this paper aims in to investigate the global behavior of solutions of eq (1), and to establish the existence of a unique solution that satisfies the give initial conditions* $y_0 = a_0,\ y_{-1} = a_{-1}$.

Then clearly for $n \geq 0$*, the initial value problem (1) possesses a positive solution.*

Now, *we will review some definitions of Stability, see* (Zhang, SHI, & GAI, 2001), (Ladas & Kocic, 2013)

Definition 1: An equilibrium points $\bar{y}$ *of equation (1) is called Locally stable if, for every* $\delta > 0$*, there exists* $\varepsilon > 0$ *such that*

$|y_{-1}-\bar{y}|<\varepsilon$ and $|y_0-\bar{y}|<\varepsilon$ implies that $|y_n-\bar{y}|<\delta$ for all $n\geq 0$. Otherwise $\bar{y}$ is said to be unstable.

Definition 2: An equilibrium points $\bar{y}$ of equation (1) is called Locally asymptotically stable if it is locally stable and there exists $\varepsilon>0$ such that $|y_{-1}-\bar{y}|<\varepsilon$ and $|y_0-\bar{y}|<\varepsilon$ implies that $\lim_{n\to\infty}|y_n-\bar{y}|=0$.

Definition 3: An equilibrium points $\bar{y}$ of equation (1) is called a global attractor if $\lim_{n\to\infty} y_n=\bar{y}$ for all $y_0, y_{-1}\in(0,\infty)$.

Definition 4: An equilibrium points $\bar{y}$ of equation (1) is called globally asymptotically stable if $\bar{y}$ is both locally asymptotically stable and a global attractor.

Usually we can be solved eq (1) by converted into a linear equation,

We let

$$x_n=\frac{1}{y_n}$$

We give as:

$$\frac{1}{x_{n+1}}=\frac{b/x_n}{1-a/x_n}=\frac{b}{x_n-a}$$

Then

$$bx_{n+1}-x_n+a=0 \qquad (3)$$

Equation (3) is asymptotically stable, the characteristic equation is

$$b\lambda^{k+1}-\lambda^k+a=0$$

(4)

Equation (1) has the unique positive equilibrium when $0 < 4ab \leq 1$, *namely*

$$\bar{y} = \frac{1 + \sqrt{1 - 4ab}}{2b}$$

Theorem: Assume that (2) holds. Then the unique positive equilibrium $\bar{y}$ *of eq (1) is global asymptotically stable provided that one of the following conditions is satisfied:*

1. $b < 1$
2. $b \geq 1$ *and* $b \geq 0$
3. $b \geq 1$ *and* $b \leq 2(b+1)$
4. $(1 - b)\bar{y} + 1 > 0$

2.*SOME LEMMAS*

In this section, we present some lemmas that will be used in the analysis of eq (1), see (Zhang, SHI, & GAI, 2001), (Ladas & Kocic, 2013)

Lemma 1: Consider the equation

$$y_{n+1} = y_n f(y_n) \tag{5}$$

assume that

$F = F(u_0, u_1)$ *is a* C^1*- function and let* $\bar{y}$ *be an equilibrium of eq (5) if all roots of the polynomial equation (4) lie in the open disk* $|\lambda| < 1$, *then the equilibrium* $\bar{y}$ *of eq (5) is asymptotic stable.*

Lemma 2: The equation

$$\mu(\lambda) = a_0\lambda^2 + a_1\lambda + a_2 = 0 \tag{6}$$

Has all its roots in the open unit disk $|\lambda| < 1$ *if and only if the equation*

$$\mu(y-1)/_{y-1} = 0$$

Has all its roots in the left- half plane $Re(y) < 0$.

Lemma 3: let $\mathcal{H}$ *be a nonincreasing function such that* $\mathcal{H} \in [[o,\infty),(0,\infty)]$ *and let* $\bar{y}$ *denoted the unique fixed point of* $\mathcal{H}$. *Then the following statements are equivalent:*

1. $\bar{y}$ *is the only fixed point of* $\mathcal{H}^2$ *in* $(0,\infty)$.
2. $\bar{y}$ *is a global attractor of all positive solutions of the recurrence relation:*

$$y_{n+1} = \mathcal{H}(y_n)\ ,\ \ n = 0,1,2, \dots \dots \text{ and } y_0 \in [0,\infty)$$

Lemma 4: Assume that the function f *in eq (5) satisfies the following hypotheses:*

1) $f \in C[(0,\infty)\times(o,\infty),(o,\infty)]$ *and* $g \in C\big[[0,\infty),(0,\infty)\big]$ *such that*
   $g(u_0,u_1) = u_0 f(u_0,u_1)$ *for* $u_0 \in (0,\infty), u_1 \in [0,\infty)$ *and*
   $g(0,u_1) = \lim\limits_{u_0\to 0} g(u_0,u_1)$
2) $f(u_0,u_1)$ *is nonincrasing in* u_0, u_1.
3) *The equation* $f(y,y) = 1$ *has a unique positive solution* $\bar{y}$
4) $f(u_0,u_1)$ *does not depend on* u_0 *or for every* $y > 0$ *and* $u \geq 0$

$$[f(y,u) - f(\bar{y},u)](y-\bar{y}) < 0$$

With

$$[f(y,\bar{y}) - f(\bar{y},\bar{y})](y-\bar{y}) < 0 \ \text{ for } y \neq \bar{y}$$

Now, define a new function

$$F(y) = \begin{cases} \max_{y \le z \le \bar{y}} g(y,z) & ,0 \le y \le \bar{y} \\ \min_{\bar{y} \le z \le y} g(x,z) & ,y > \bar{y} \end{cases}$$

(7)

Where

$$g(y,z) = zf(z,y)f(\bar{y},z)$$

(8)

then

a) $F \in C[(0,\infty),(0,\infty)]$ *and* F *is nonincreasing in* $[0,\infty)$.
b) *Suppose that* F *has no periodic points of prime period 2. then* $\bar{y}$ *is a global attractor of all positive solutions of eq (5).*

3. ***MAIN RESULTS***

Theorem 1: Assume that $0 \le b < 1$, *let* $\{y_n\}$ *be a positive solution of eq (1) then* $\lim_{n\to\infty} y_n = \bar{y}$ *where* $\bar{y}$ *is the unique positive equilibrium of eq (1)*

Further, if and only if $(y_{-1}, y_0) \neq (\bar{y}, \bar{y})$ *then the semi cycles of every positive solution of eq (1) have length 2.*

Proof

To analyze the give recurrence relation, we start by finding the equilibrium point. The equilibrium point $\bar{y}$ *satisfies* $y_{n+1} = y_n = \bar{y}$ *substituting* $\bar{y}$ *into the recurrence relation gives* $\bar{y} = \frac{b\bar{y}}{1-a\bar{y}}$

Rearranging terms yields: $a\bar{y}^2 - \bar{y}(1-b) = 0,\ 1 - a\bar{y} \neq 0$

Thus, the solutions are $\bar{y} = 0$ *or* $a\bar{y} = 1 - b$

Since we are only interested in the positive solution, we have:

$$\bar{y} = \frac{1-b}{a} \quad .$$

For $0 \leq b < 1$ *and* $> 0, \bar{y} > 0$, *making it the unique positive equilibrium of the recurrence relation.*

Next, we analyze the stability of $\bar{y}$ *by examining the derivative of the recurrence function:*

$f(y) = \frac{by}{1-ay}$. , $f'(y) = \frac{b}{(1-ay)^2}$

At $y = \bar{y}$, *we compute:* $f'(y) = \frac{b}{(1-a\bar{y})^2}$.

Substituting $\bar{y} = \frac{1-b}{a}$, *we find* $f'(\bar{y}) = \frac{1}{b}$

Since $0 < b < 1$, *it follows that* $|f'(\bar{y})| > 1$, *meaning* $\bar{y}$ *is unstable. The sequence does not converge to* $\bar{y}$, *instead, it exhibits oscillatory behavior.*

To determine the periodicity, consider two successive terms of the sequence:

$y_0 = \frac{by_{-1}}{1-ay_{-1}}$, $y_1 = \frac{by_0}{1-ay_0}$.

For a periodic cycle of length 2.

We require $y_1 = y_{-1}$.

Substituting y_0 *into* y_1:

$$y_1 = \frac{b^2 y_{-1}}{(1 - ay_{-1})^2}.$$

Equating $y_1 = y_{-1}$: $y_{-1} = \frac{by_{-1}}{1-ay_{-1}}$ $, y_{-1} > 0$

$$1 = \frac{b^2}{(1 - ay_{-1})^2}$$

Taking the square root:

$1 - ay = \pm b$, *since* $0 < b < 1$, *we take the positive root:*

$$1 - ay_{-1} = b$$

Solving for y_{-1}: $\quad y_{-1} = \frac{1-b}{a}$.

This confirms that if $(y_{-1}, y_0) = (\bar{y}, \bar{y})$, *the sequence remains constant. Otherwise, the sequence oscillates with a period of 2.*

Thus, we conclude that, the unique positive equilibrium of the recurrence relation is $\bar{y} = \frac{1-b}{a}$.

Every positive solution of the recurrence relation exhibits semi cycles of length 2 if and only if $(y_{-1}, y_0) \neq (\bar{y}, \bar{y})$.

Theorem 2: Suppose that (2) holds, then the unique positive equilibrium $\bar{y}$ *of eq (1) is a global attractor of all positive solution of (1).*

Proof

Eq (1) can be rewritten as follows

$y_{n+1} = y_n \frac{b}{1-ay_n}$,

Set

$$f(u_0, u_1) = \frac{b}{1 - au_0}$$

And

$$g(u_0, u_1) = u_0 f(u_0, u_1)$$

It's straightforward to verify that the functions f *and* g *satisfy the hypotheses of lemma (4).*

Moreover, the function g *as defined by (8) takes the form*

$$g(y,z) = zf(z,y)f(\bar{y},z)$$

$$= z\frac{b}{1+az}\frac{b}{1+a\bar{y}}$$

By (7) we will construct the function F

Since

$$\frac{d}{dz}\left(\frac{bz}{1-az}\right) = \frac{b}{(1-az)^2}$$

The function is increasing and

$$\max_{y\le z\le \bar{y}}\left(\frac{bz}{1-az}\right) = \frac{b\bar{y}}{1-a\bar{y}} = \bar{y}$$

Also

$$\min_{\bar{y}\le z\le \bar{y}}\left(\frac{bz}{1-az}\right) = \frac{b\bar{y}}{1-a\bar{y}} = \bar{y}$$

Hence the function F *is given by*

$$\mathrm{F}(y) = \bar{y}\left(\frac{b}{1-ay}\right)$$

To complete this proof, it remains to show that F *has no period 2. then*

$$\mathrm{F}(y) = \frac{b\bar{y}}{1-ay}$$

The only solution of the system

$z = \frac{b\bar{y}}{1-ay}, y = \frac{b\bar{y}}{1-az}$ *(9)*

is $\bar{y} = y = z.$

On the contrary assume that there exists a solution $\{y, z\}$ *of the system (9) such that* $0 < y < \bar{y} < z$.

It is straightforward to easy to see that $y = \mathrm{F}(z)$, *and* $z = \mathrm{F}(y)$ *from (9) such that*

$$z - ayz = y - ayz$$

Therefore, $= z$ *. this is a contradiction. from lemma (4) we have the result is established.*

5. Conclusions

. In this paper we gave the definitions of stability, presented some lemmas which will be used in the eq (1) and established that the unique positive equilibrium $\bar{y}$ *of eq (1) is a global attractor of all positive solution of (1).*

REFERENCES

[1] Li Wantong, Zhang Yanhong, Su Youhui. (2005). Global Attractivity in A Class of Higher-Order Nonlinear Difference Equations. Proceedings NNSF of China, 25B (1), 59-66.

[2] Koci´C, V. L., & Ladas, G. (1992). Global Attractivity in Nonlinear Delay Difference Equations. Proceedings of the American Mathematical Society, 115(4), 1083-1088

[3] D. C. Zhang, B.SHI &M.J.GAI (2001). A Rational Recursive Sequence. Computers and Mathematics with Applications 41:(2001) 301-306.

[4] E. Camouzis, G. Ladas, L. W. Rodrigues & S. North shield. The Rational Recursive Sequence $x_{n+1} = {(\beta x_n^2)}/{(1 + x_{n-1}^2)}$ *. Computers Math. Applic Vol. 28, No.1-3, pp. 37-43,1994.*

[5] Kocic, V., Ladas, G. (2013). Global Behavior of Nonlinear Difference Equations of Higher Order with Applications. Netherlands: Springer Netherlands.

[6] M.T. Aboutaleb, M.A. El-Sayed, & A.E. Hamza. Stability of recursive sequence $x_{n+1} = \frac{a-bx_n}{A+x_{n-1}}$, *Journal of Mathematical Analysis and Applications, vol. 261. no.1, pp.126-133,2001.*

[7] L. Clemente, M. Donten- Bury, F. Mazowiecki &M. Pilipczuk. On Rational Recursive Sequences.40th International Symposium on Theoretical Aspects of Computer Science (Stacs 2023) Schloss-Dagstuhl-Leibniz Zentrum Fur Informatik,2023.