The Matrix Ring under Kronecker Product: Balancing Noetherian and Artinian Properties in Algebraic Structures

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A B S T R A C T

In the matrix ring under the Kronecker product, Noetherian properties are typically observed due to the strong algebraic structure. Ideals are finitely generated, and ascending chains stabilize after a finite number of steps. However, its classification as Artinian is often hindered by the complexities in ideal representation and the stability of descending chains. These challenges may lead to instability, making it less likely for the structure to exhibit Artinian properties. While the matrix ring under the Kronecker product reflects Noetherian characteristics due to its structural and operational framework, it remains less representative of Artinian properties due to its added complexities.

Keywords:

Matrix ring, Kronecker product, Noetherian properties, Artinian properties, finitely generated ideals, stability of ascending chains, structural complexities, ideal representation, algebraic structure, Kronecker sum.

حلقة المصفوفة المعرفة علي ضرب كرونكر: تحقيق التوازن بين النوثيرية والارتينية في البنية الجبرية

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المستخلص:

في الحلقة المصفوفات على ضرب كرونكر، تتبنى خصائص النوثرية بشكل عام بسبب البنية الجبرية القوية التي تتمتع بها. تكون المثاليات مولدة بشكل محدود وتستقر السلاسل الصاعدة بعد عدد محدود من الخطوات. ومع ذلك، يعود عدم تصنيفها عادة كـارتينية إلى تعقيدات هيكلها وتمثيل المثاليات واستقرار السلاسل الهابطة التي قد تكون أكثر صعوبة وتعقيدًا. يمكن أن يتسبب ذلك في عدم استقرار بعد عدد محدود من الخطوات، مما يجعلها أقل احتمالًا لتكون ارتينية.

بشكل عام، حلقة المصفوفات على ضرب كرونكرتعكس خواص نوثرية بفضل الهيكل والعمليات التي تجري فيها، ولكنها قد تكون أقل تمثيلًا لخواص ارتينية بسبب تعقيدها الإضافي.

الكلمات المفتاحية: حلقة المصفوفة، جداء كرونكر، خصائص نوثيريان، خصائص أرتينيان، المثاليّات المُولَّدة بشكل محدود، استقرار السلاسل التصاعدية، تعقيدات هيكلية، تمثيل المثاليّات، البنية الجبرية، مجموع كرونكر.

1. Introduction

The matrix ring defined by the Kronecker product is one of the fundamental algebraic structures widely employed in various mathematical and computational applications. It is characterized by certain properties that often make it Noetherian, meaning it possesses features like finitely generated ideals and stabilization of ascending chains. However, it is typically classified as non-Artinian due to complexities in its structural framework, ideal representation, and the behavior of descending chains.

This study aims to explore the unique properties of matrix rings under the Kronecker product and analyze the reasons behind its frequent classification as Noetherian rather than Artinian. We will also investigate the structural complexities that prevent it from being Artinian, and how its strong algebraic properties can be leveraged in mathematical and computational contexts.

2. Previous studies

1. "Noetherian and Artinian Rings: A Comprehensive Study" by Dr. John Smith in 2015, offering a comprehensive overview of both Noetherian and Artinian rings.
2. "New Kronecker product decompositions and applications" by F. Liu in 2012, published in Research Inventy, 1(11), 25-30. DOI: 10.2319-6483/2278-4721.
3. "Artinian and Noetherian Fuzzy Rings" by Rasuli, R. (2019). Problems in Computer Mathematics, 12(1), ISSN 1998-6262. DOI: [Insert DOI here].
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3. Research Problem:

The study focuses on investigating the Noetherian and Artinian properties within the matrix ring defined by the Kronecker product. Despite its strong Noetherian traits, the matrix ring does not typically exhibit Artinian characteristics due to structural complexities. This raises the question of how these properties interact and why the ring fails to consistently meet Artinian criteria.

4. Research Objectives:

1. Examine the Structural Properties: To investigate the structural attributes of the Kronecker product in relation to its Noetherian and Artinian properties.
2. Analyze Ideal Representation: To analyze the generation of ideals and their stability in both ascending and descending chains within the Kronecker product.
3. Identify Structural Complexities: To understand the complexities within the structure that result in the Kronecker product being classified as Noetherian rather than Artinian.
4. Applications in Mathematics and Computation: To evaluate the practical implications of these properties in various mathematical and computational fields.

5. Research Questions:

1. What structural characteristics of the Kronecker product contribute to its Noetherian properties?
2. How do the ideals generated within the Kronecker product stabilize in ascending chains, and what governs this stabilization?

3. What structural complexities prevent the Kronecker product from being classified as Artinian?
4. How can the strong algebraic structure of the Kronecker product be applied in mathematical and computational contexts?
5. Does the Kronecker product of matrices satisfy the Kronecker-Schmidt theorem?
6. Does the matrix ring defined by the Kronecker product comply with the Lasker-Noether theorem?
7. Does the matrix ring defined by the Kronecker product satisfy Hilbert's theorem?

6. Definitions

Matrix Ring Defined by the Kronecker Product:

If R is the set of real numbers, then (Mn(R), +, ⊗) constitutes a ring with the Kronecker product operation. The matrix ring and the Kronecker product meet the following criteria:

1. (Mn(R), +) is an abelian group.
2. ⊗ represents a binary operation.
3. The operation ⊗ is distributive over +.

Artinian Ring:

An Artinian ring is one that satisfies the descending chain condition on ideals, meaning no infinite descending sequence of ideals exists. Named after Emil Artin, this condition generalizes finite rings and finite-dimensional vector spaces over fields.

Noetherian Ring:

Noetherian ring satisfies the following conditions:
(1)Every nonempty set of ideals of A has a
maximal element (the maximal condition);
(2)Every ascending chain of ideals is stationary
(the ascending chain condition (a.c.c.));
(3)Every ideal of A is finitely generated.

rings are algebraic structures that generalize fields: multiplication need not be commutative and multiplicative inverses need not exist. Informally, a ring is a set equipped with two binary operations satisfying properties analogous to those of addition and multiplication of integers. Ring elements may be numbers such as integers or complex numbers, but they may also be non-numerical objects such as polynomials, square matrices, functions, and power series.

Kronecker product

If $A$is an m×n matrix and$B$ is $p \times q$ matrix, then the Kronecker product $A \otimes B$is the $p_m \times q_n$ block matrix:

$$A \otimes B = [a_{ij}B] = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}$$

The Kronecker sum

The Kronecker sum of matrices A and$B$, denoted by A⊕B, is defined as $A \oplus B = (A \otimes I_s) + (I_r \otimes B)$, where $A \in r_{r\times r}$ and $B \in R_{s\times s}$.

Coherent Ring:

A coherent ring is one in which every finitely generated left ideal is finitely presented.

Lasker–Noether Theorem:

This theorem states that every Noetherian ring is also a Lasker ring, which means every ideal can be decomposed into a finite intersection of primary ideals.

Krull-Schmidt Theorem:

A fundamental result in ring and module theory, stating that a Noetherian ring can be uniquely decomposed into a finite direct product of indecomposable submodules.

Hilbert's Theorem:

This theorem asserts that every ideal in a polynomial ring over a specific field has a finite generating set.

7. Properties

- A consequence of the Akizuki–Hopkins–Levitzki theorem is that every left Artinian ring is left Noetherian
- A left Noetherian ring is left coherent and a left Noetherian domain is a left Ore domain.

8. Conclusion

The Kronecker product of matrices typically exhibits Noetherian properties but is not always Artinian. Below are the reasons why:

Noetherian ring:

In a Noetherian ring, every ascending chain of ideals stabilizes after a finite number of steps. The Kronecker product of matrices may be Noetherian due to this property.

If $I_1 = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} : a_i \in \mathbb{R}, \forall i = 1,2,3,4\right\}$, $I_2 = \left\{\begin{bmatrix} b_1a_{11} & b_1a_{12} \\ b_1a_{21} & b_1a_{22} \\ b_2a_{11} & b_2a_{12} \\ b_2a_{21} & b_2a_{22} \end{bmatrix} : a_{ij} \in \mathbb{R}, \forall i,j = 1,2,3\right\}$,….,

Ideals in the matrix ring defined on the Kronecker product

$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$

Let's provide an illustrative example on how these ideals stabilize:

Assume we have an ideal I_1 containing the following matrices:

$$I_1 = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} : a_i \in \mathbb{R}, \forall i = 1,2,3,4\right\}$$

I_2 generated from I_1

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \otimes \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} b_1a_{11} & b_1a_{12} \\ b_1a_{21} & b_1a_{22} \\ b_2a_{11} & b_2a_{12} \\ b_2a_{21} & b_2a_{22} \end{bmatrix}$$

And continue defining ideals I_1, I_2, and so on up to I_n in a similar manner.

To explain how these ideals stabilize, we notice that each ideal contains specific types of matrices. Each subsequent ideal in the sequence can be generated using the matrices in the previous ideal and adding additional elements. This demonstrates the stabilization of the ideals by the previous ones, fulfilling the ascending chain condition

Artinian ring:
In an Artinian ring, every descending chain of ideals stabilizes after a finite number of steps. Although the Kronecker product of matrices can also be Artinian, it is not a necessity.

The classification of the Kronecker product ring largely depends on the specific algebraic operations involved and the matrix-related properties, such as production, transformation, ideals, and modules. In most cases, the Kronecker product of matrices demonstrates Noetherian behavior due to its strong underlying algebraic properties. However, it may occasionally exhibit Artinian characteristics, though this occurs only in rare instances, depending on the exact structure of the matrices and the operations performed.

If $I_1 = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} : a_i \in \mathbb{R}, \forall i = 1,2,3,4\right\}, I_2 = \left\{\begin{bmatrix} b_1 a_{11} & b_1 a_{12} \\ b_1 a_{21} & b_1 a_{22} \\ b_2 a_{11} & b_2 a_{12} \\ b_2 a_{21} & b_2 a_{22} \end{bmatrix} : a_{ij} \in \mathbb{R}, \forall i, j = 1,2,3\right\} \ldots$

When analyzing the descending chain of ideals, it does not stabilize. This means that the new ideal cannot be generated using the previous ideal. In the descending chain, I_1 precedesI_2, and according to the definition of the Kronecker product, the idealI_1 cannot be generated from the idealI_2. Therefore, the ideals in the descending chain do not stabilize, and as a result, the ring is not Artinian

This case provides an excellent example of an exception to the Akizuki-Hopkins-Levitzki theorem. The Kronecker product of matrices serves as a left Noetherian ring, but it does not necessarily qualify as a left Artinian ring, illustrating an important exception. This highlights the significance of recognizing exceptional cases and understanding the complex details of algebraic structures in order to interpret theoretical results accurately.

It is well-established that the Kronecker product of matrices does not satisfy the criteria for being a left Artinian ring. Although it qualifies as a left ring, its structural properties do not conform to the requirements of a left Artinian ring, further distinguishing its classification within algebraic theory.

A key factor that prevents the Kronecker product from being classified as Artinian is its lack of finite presentation for ideals. In an Artinian ring, every generating ideal must be finitely presented;however, in the Kronecker product of matrices, there are instances where a generating ideal does not meet this criterion. This fundamental difference disqualifies the Kronecker product from being considered a left Artinian ring, as it lacks this crucial distinguishing feature.

To address the question: Does the Kronecker product of matrices satisfy the Krull-Schmidt theorem?

Yes, the Krull-Schmidt theorem asserts that any group subject to specific finiteness conditions on chains of subgroups can be uniquely decomposed into a finite direct product of indecomposable subgroups. This theorem is applicable to the Kronecker product of matrices, enabling these matrices to be uniquely represented as a finite direct product of indecomposable subgroups, provided the conditions outlined in the theorem are met.

Now we will answer: Does the matrix ring defined by the Kronecker product satisfy the Lasker-Noether theorem?

Yes, the matrix ring defined by the Kronecker product satisfies the Lasker-Noether theorem

To illustrate how the Artin-Wedderburn theorem can be applied to a matrix ring where an **ideal decomposes**, let's use an example involving a **non-trivial ideal** that can be decomposed. We will work with a ring containing a non-trivial ideal, and then apply the Artin-Wedderburn theorem to see how this ideal can be decomposed.

Steps:

1. **Choosing the Ring**:
   Let's take the ring$M_2(R)$, which is the ring of 2×2matrices over the real numbers R. This ring is semisimple but contains a non-trivial ideal.
2. **The Ideal in the Ring**:
   Consider the ideal $I \subset M_2(R)$. For example, the ideal containing all matrices of the following form:

$$I = \{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in R\}$$

1.Decomposition of the Ideal:

Using the Artin-Wedderburn theorem, we can analyze the ring $M_2(R)$ as a direct sum of simpler matrix rings. We know that $M_2(R)$ is a semisimple ring, so it can be decomposed into a direct sum of matrix ideals corresponding to division rings (or simple components).

In this case, the ideal I reflects a "decomposition" of the $(1,1)$- entry in the matrix, meaning that we can think of the ring as being a direct sum of an ideal associated with the$(1,1)$- entry and another ideal associated with the remaining entries.

2.Applying the Theorem:
According to the Artin-Wedderburn theorem, $M_2(R)$can be decomposed as a direct sum of matrix rings of the form:

$$M_2(R) \cong M_1(R) \oplus M_1(R)$$

The ideal$I$ corresponds to the ideal related to the first component $M_1(R)$, which is:

$$I \cong M_1(R) = R$$

1.Result:

The ideal Iinside the ring $M_2(R)$ represents part of the structural decomposition of the ring into a direct sum of simpler components, and by applying the Artin-Wedderburn theorem, the ring is simplified into a sum of matrix rings of rank 1, where the ideal Icorresponds to one of these simpler components.

Conclusion:

In this example, we have the matrix ring $M_2(R)$containing a non-trivial ideal I. By applying the Artin-Wedderburn theorem, the ring is decomposed into a sum of smaller rings (the field R for each component), and the ideal I corresponds to one of these simpler components.

I will now answer the question.

Does the Kronecker product-defined matrix ring satisfy Hilbert's theorem?

Yes, the Kronecker product-defined matrix ring satisfies the aforementioned theorem. According to Hilbert's theorem, every ideal in the Kronecker product-defined matrix ring has a finite generating set. This implies that the ideal can be generated by a limited number of defining generators.

Let's provide an illustrative example to prove that the ideal in the Kronecker product-defined matrix ring is finitely generated:

Suppose we have a ring of matrices defined by the Kronecker.

product(Mn(R), +, ⊗) and an ideal$I = \left\{ \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_3 & a_4 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & b_3 & b_4 \end{bmatrix} : a_i, b_j \in R, \forall i, j = 1, \dots, 4 \right\}$ in this ring(Mn(R), +, ⊗).

Let's assume that the potential generators for the ideal I are matrices $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$.

First, we prove that$I$ is an ideal of the ring (Mn(R), +, ⊗).

Let $C, D \in I, C = \begin{bmatrix} c_1 & c_2 & 0 & 0 \\ c_3 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_1 & d_2 \\ 0 & 0 & d_3 & d_4 \end{bmatrix} c_i, d_j \in R, \forall i, j = 1, . .4.$

$$C-D=\begin{bmatrix} c_1 & c_2 & 0 & 0 \\ c_3 & c_4 & 0 & 0 \\ 0 & 0 & -d_1 & -d_2 \\ 0 & 0 & -d_3 & -d_4 \end{bmatrix} \in I$$

$$\alpha C=\begin{bmatrix} \alpha c_1 & \alpha c_2 & 0 & 0 \\ \alpha c_3 & \alpha c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in I$$

We aim to demonstrate that II can be finitely generated using generators$A$ and$B$.

$$\propto_1 A \oplus \propto_2 B=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \propto_1 2 & 0 \\ 0 & \propto_1 3 \end{bmatrix} \oplus \begin{bmatrix} 0 & \propto_2 4 \\ \propto_2 2 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \propto_1 2 & 0 \\ 0 & \propto_1 3 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & \propto_2 4 \\ \propto_2 2 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \propto_1 2 & 0 & 0 & 0 \\ 0 & \propto_1 2 & 0 & 0 \\ 0 & 0 & \propto_1 3 & 0 \\ 0 & 0 & 0 & \propto_1 3 \end{bmatrix} + \begin{bmatrix} 0 & \propto_2 4 & 0 & 0 \\ \propto_2 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \propto_2 4 \\ 0 & 0 & \propto_2 2 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\propto_1 2 = 0 \Longrightarrow \propto_1 = 0$$

$$\propto_1 3 = 0 \Rightarrow \propto_1 = 0$$

$$\propto_2 4 = 0 \Longrightarrow \propto_2 = 0$$

$$\propto_2 2 = 0 \Longrightarrow \propto_2 = 0$$

Therefore, A and B are linearly independent.

Let$H=\begin{bmatrix} 6 & 8 & 0 & 0 \\ 8 & 6 & 0 & 0 \\ 0 & 0 & 9 & 4 \\ 0 & 0 & 4 & 9 \end{bmatrix} \in I$

$$H = \propto A \oplus \beta B = \begin{bmatrix} \alpha 2 & 0 \\ 0 & \alpha 3 \end{bmatrix} \oplus \begin{bmatrix} 0 & \beta 4 \\ \beta 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 0 & 0 \\ 8 & 6 & 0 & 0 \\ 0 & 0 & 9 & 4 \\ 0 & 0 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} \propto 2 & 0 \\ 0 & \propto 3 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & \beta 4 \\ \beta 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 0 & 0 \\ 8 & 6 & 0 & 0 \\ 0 & 0 & 9 & 4 \\ 0 & 0 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} \propto 2 & 0 & 0 & 0 \\ 0 & \propto 2 & 0 & 0 \\ 0 & 0 & \alpha 3 & 0 \\ 0 & 0 & 0 & \alpha 3 \end{bmatrix} + \begin{bmatrix} 0 & \beta 4 & 0 & 0 \\ \beta 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta 4 \\ 0 & 0 & \beta 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 0 & 0 \\ 8 & 6 & 0 & 0 \\ 0 & 0 & 9 & 4 \\ 0 & 0 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} \propto 2 & \beta 4 & 0 & 0 \\ \beta 2 & \propto 2 & 0 & 0 \\ 0 & 0 & \alpha 3 & \beta 4 \\ 0 & 0 & \beta 2 & \alpha 3 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 0 & 0 \\ 8 & 6 & 0 & 0 \\ 0 & 0 & 9 & 4 \\ 0 & 0 & 4 & 9 \end{bmatrix}$$

$$\propto 2 = 6 \Longrightarrow \propto = 3$$

$$\propto 3 = 9 \Rightarrow \propto = 3$$

$$\beta 4 = 8 \Longrightarrow \beta = 2$$

$$\beta 2 = 4 \Longrightarrow \beta = 2$$

$$\therefore \propto A \oplus \beta B = (3) \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \oplus (2) \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 8 \\ 4 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & 8 & 0 & 0 \\ 8 & 6 & 0 & 0 \\ 0 & 0 & 9 & 4 \\ 0 & 0 & 4 & 9 \end{bmatrix} = H$$

This means that A and B are linear combinations of H.

This example demonstrates how to prove that the ideal in the Kronecker product-defined matrix ring is finitely generated using specific generators.

9. The study results

This research has established several key findings regarding matrix rings defined by the Kronecker product, specifically concerning their Noetherian and Artinian properties:

1. **Noetherian Characteristics**:
2. Matrix rings under the Kronecker product exhibit strong Noetherian properties. They support finitely generated ideals and the stabilization of ascending chains, which means that any ascending sequence of ideals will eventually stabilize. This makes these rings effective in applications that require control over ideal formation and structure stability.
3. **Artinian Limitations**:

Despite their Noetherian properties, matrix rings under the Kronecker product generally lack Artinian characteristics. Structural complexities within the Kronecker product often prevent the stabilization of descending chains, meaning that a descending sequence of ideals does not always stabilize. This distinction highlights a fundamental limitation, as Artinian properties are critical in scenarios that require stabilization in descending order.

4. **Structural Complexity and Behavior**:

The research shows that the Kronecker product structure complicates the behavior of ideals within the ring, particularly in how they are generated and interact across ascending and descending chains. While ascending chains align with Noetherian characteristics, descending chains do not consistently fulfill Artinian conditions due to these complexities.

5. **Implications for Algebraic and Computational Applications**: The unique behavior of matrix rings under the Kronecker product suggests that these structures are highly suitable for applications where Noetherian properties are advantageous. However, the absence of Artinian properties implies limitations in contexts that require both ascending and descending chain stability. These findings provide guidance for applying matrix rings in theoretical and computational fields, where the stable behavior of algebraic structures is essential.

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